

Orbit measures and interlaced determinantal point processes

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Abstract

We study random interlaced configurations on $\mathbb{N} \times \mathbb{R}$ considering the eigenvalues of the main minors of Hermitian random matrices of the classical complex Lie algebras. We show that these random configurations are determinantal and give their correlation kernels.

1 Introduction

This note announces the results of (4). Baryshnikov (1) has studied the law of the eigenvalues of the main minors of a random matrix from the GUE, developing the connexions with uniform measure on Gelfand Cetlin patterns. In this paper we establish such connexions in a more general context, using a version of Heckman's theorem (6). From this theorem and from classical branching rules, we deduce the joint law of the eigenvalues of the main minors of random Hermitian matrices of classical complex Lie algebras, with invariant law under some unitary transformations. This allows us to show that the associated point processes are determinantal. For the GUE minors process, this has been proved by Johansson and Noordenstam (7) and Okounkov and Reshetikhin (8) (see also, very recently, Forrester and Nagao (5)). Thus, we generalise some of their results to all the classical complex Lie algebras.

2 Approximation of orbit measures

Let G be a connected compact Lie group with Lie algebra \mathfrak{g} and complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We choose a maximal torus T of G and we denote by \mathfrak{t} its Lie algebra. We equip \mathfrak{g} with an $\text{Ad}(G)$ -invariant inner product $\langle \cdot, \cdot \rangle$ which induces a linear isomorphism between \mathfrak{g} and its dual \mathfrak{g}^* and intertwines the adjoint and the coadjoint action of G . We consider the roots system R , i.e. all $\alpha \in \mathfrak{t}^*$ such that there is a non zero $X \in \mathfrak{g}_{\mathbb{C}}$ such that for all $H \in \mathfrak{t}$, $[H, X] = i\alpha(H)X$. We choose the set Σ of simple roots of R , $C(G) = \{\lambda \in \mathfrak{t}^* : \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Sigma\}$ the corresponding Weyl chamber and $P^+(G) = \{\lambda \in \mathfrak{t}^* : 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N}, \text{ for all } \alpha \in \Sigma_G\}$ the corresponding set of integral dominant weights.

We consider a connected compact subgroup H of G with Lie algebra \mathfrak{h} . We can choose a maximal torus S of H contained in T and a corresponding set of integral dominant weights denoted by $P^+(H)$. For $x \in \mathfrak{g}^*$, let $\pi_H(x)$ be the orthogonal projection of x on \mathfrak{h}^* .

For $\lambda \in P^+(G)$, we denote by V_λ the irreducible \mathfrak{g} -module with highest weight λ and $\dim_G(\lambda)$ the dimension of V_λ . For $\beta \in P^+(H)$ we denote by $m_H^\lambda(\beta)$ the multiplicity of the \mathfrak{h} -module with highest weight β in the decomposition into irreducible components of V_λ considered as an \mathfrak{h} -module. Rules giving the value of the multiplicities m_H^λ are called *branching rules*.

The intersection between the orbit of an element $x \in \mathfrak{g}^*$ under the coadjoint action of G and the closure $\bar{C}(G)$ contains a single point that we call the radial part of x and denote by $r_G(x)$. The same holds for H and, for $x \in \mathfrak{g}^*$, we write $r_H(x)$ instead of $r_H(\pi_H(x))$.

The following proposition is a version of Heckman's theorem (6) on asymptotic behaviour of multiplicities. As usual δ_x is the Dirac measure at x .

Proposition 1. *Let $x \in \bar{C}(G)$. Let $(\lambda_n)_{n \geq 0}$ be a sequence of elements in $P^+(G)$ and $(\epsilon_n)_{n \geq 0}$ a sequence of positive real numbers such that ϵ_n converges to zero and $\epsilon_n \lambda_n$ converges to x , as n tends to $+\infty$. Then the sequence $(\mu_n)_{n \geq 0}$ of probability measures on $\bar{C}(H)$ defined by*

$$\mu_n = \sum_{\beta \in P^+(H)} \frac{\dim_H(\beta)}{\dim_G(\lambda_n)} m_H^{\lambda_n}(\beta) \delta_{\epsilon_n \beta}.$$

converges to the law of $r_H(Ad(g)x)$, with g distributed according to the normalised Haar measure on G .

We deduce from this proposition and from classical branching rules the descriptions of convolutions or projections of $Ad(G)$ -invariant measures on classical Lie algebras. Let us give a first application when G is the symplectic unitary group $Sp(n)$.

Example 1. *This example concerns random variables which play the same role for $Sp(n)$ as the Laguerre ensemble for $U(n)$. Let $(X_i)_{i \geq 1}, (Y_i)_{i \geq 1}$ be two independent sequences of random variables in \mathbb{C}^n with independent standard complex Gaussian components. We consider $M_k = \sum_{i=1}^k S(X_i, Y_i) R(X_i, Y_i)^*$, $k \geq 1$, where, for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{C}^n$,*

$$S(x, y) = \begin{pmatrix} s(x_1, y_1) \\ \vdots \\ s(x_n, y_n) \end{pmatrix}, R(x, y) = \begin{pmatrix} r(x_1, y_1) \\ \vdots \\ r(x_n, y_n) \end{pmatrix}, s(a, b) = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}, r(a, b) = \begin{pmatrix} a & b \\ \bar{b} & -\bar{a} \end{pmatrix}, a, b \in \mathbb{C}.$$

Let $\Lambda^{(k)}$ be the positive eigenvalues of M_k . The process $(\Lambda^{(k)})_{k \geq 1}$ is an inhomogeneous Markovian process. Its transition kernel is deduced from Proposition 1 considering tensor product by irreducible representations with highest weight proportional to the highest weight of the standard representation (see (4)).

3 Eigenvalues of the main minors of classical Hermitian matrices

3.1 Classical Hermitian matrices

We study a class of measures on generalised Gelfand Cetlin cones (see, e.g., Berenstein et Zelevinsky (2)). We denote by $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$) the set of $n \times n$ complex (resp. real) matrices and I_n the unit matrix of $M_n(\mathbb{C})$. We consider the classical compact groups. G_a is the unitary group $U(n) = \{M \in M_n(\mathbb{C}) : M^* M = I_n\}$, $G_b = SO(2n+1)$, G_c is the unitary symplectic group written in a somewhat unusual way as $Sp(n) = \{M \in U(2n) : M^t J M = J\}$ where all entries of $J = (J_{i,j})_{1 \leq i,j \leq 2n}$ are zero except $J_{2i,2i-1} = -J_{2i-1,2i} = 1$, $i = 1, \dots, n$, and $G_d = SO(2n)$, where $SO(k)$ is the orthogonal group $\{M \in M_k(\mathbb{R}) : M^* M = I_k, \det(M) = 1\}$. These compact groups correspond to the roots system of type A, B, C, D . We write \mathfrak{g}_ν their Lie algebras and $\mathcal{H}_\nu = i\mathfrak{g}_\nu$, $\nu = a, b, c, d$, the sets of associated Hermitian matrices.

The radial part of $M \in \mathcal{H}_\nu$ can be identified with the ordered eigenvalues of M when $\nu = a$, resp. ordered positive eigenvalues of M when $\nu = b, c$. When $\nu = d$, the Weyl chamber is $C = \{D(x) : x \in \mathbb{R}^n, x_1 > \dots > x_{n-1} > |x_n|\}$ where $D(x)$ is the matrix in $M_{2n}(\mathbb{C})$ all the entries of which are zero except $D(x)_{2k,2k-1} = -D(x)_{2k-1,2k} = ix_k$, when $k = 1, \dots, n$. For $M \in \mathcal{H}_d$ there exists a unique $x \in \mathbb{R}^n$ such that $\{k M k^*, k \in G_d\} \cap \bar{C} = \{D(x)\}$. We call it the radial part of M . These definitions of the radial part are the same as in Section 2, up to some identifications.

3.2 Gelfand Cetlin cones

For $x, y \in \mathbb{R}^r$ we write $x \succeq y$ if x and y are interlaced, i.e.

$$x_1 \geq y_1 \geq x_2 \geq \dots \geq x_r \geq y_r.$$

When $x \in \mathbb{R}^{r+1}$ we add the relation $y_r \geq x_{r+1}$. Let us consider the following Gelfand Cetlin cones:

$$\begin{aligned} GC_a &= \{x = (x^{(1)}, \dots, x^{(n)}) : x^{(i)} \in \mathbb{R}^i, x^{(i)} \succeq x^{(i-1)}, i \leq n\}; \\ GC_c &= \{x = (x^{(1)}, \dots, x^{(2n)}) : x^{(2i)}, x^{(2i-1)} \in \mathbb{R}_+^i, x^{(k)} \succeq x^{(k-1)}, i \leq n, k \leq 2n\}; \\ GC_b &= \{x = (x^{(1)}, \dots, x^{(2n)}) : x^{(2i)} \in \mathbb{R}_+^i, x^{(2i-1)} \in \mathbb{R}_+^{i-1} \times \mathbb{R}, |x| \in GC_c\}; \\ CG_d &= \{x = (x^{(1)}, \dots, x^{(2n-1)}) : \text{there exists } x^{(2n)} \in \mathbb{R}_+^n \text{ such that } (x, x^{(2n)}) \in CG_b\}. \end{aligned}$$

where, in the definition of GC_b , $|x|$ has the same components as x except $x_i^{(2i-1)}$ which is replaced by $|x_i^{(2i-1)}|$. The first line of the cone GC_ν is $x^{(n)}$ when $\nu = a$, $x^{(2n)}$ when $\nu = b, c$ and $x^{(2n-1)}$ when $\nu = d$. For each λ in \mathbb{R}^n we let $GC_\nu(\lambda)$ be the cone with first line λ .

For $M \in \mathbb{M}_n(\mathbb{C})$ the main minor of order $m \leq n$ of M is the submatrix $(M_{ij})_{1 \leq i, j \leq m}$. When $\nu = a, b, d$ the main minor of a matrix in \mathfrak{g}_ν is also in one of the classical Lie algebra (of the same type except when $\nu = b$ or d and in that case it is of type B when m is odd and of type D when m is even). This also true for when m is even when $\nu = c$. Thus we can define the radial part of a main minor of $M \in \mathcal{H}_\nu$ as above.

Definition 1. For $\nu = a, b, c, d$ let $M \in \mathcal{H}_\nu$. We write

$$\begin{aligned} \lambda_a(M) &= (\lambda_a^{(1)}(M), \dots, \lambda_a^{(n)}(M)), \lambda_b(M) = (\lambda_b^{(2)}(M), \lambda_b^{(3)}(M), \dots, \lambda_b^{(2n+1)}(M)) \\ \lambda_c(M) &= (\lambda_c^{(2)}(M), \lambda_c^{(4)}(M), \dots, \lambda_c^{(2n)}(M)), \lambda_d(M) = (\lambda_d^{(2)}(M), \lambda_d^{(3)}(M), \dots, \lambda_d^{(2n)}(M)) \end{aligned}$$

where $\lambda_\nu^{(i)}(M)$ is the radial part of the main minor of order i of M .

Using the proposition 1 we get the following Theorem.

Theorem 1. For $\nu = a, b, c, d$, let $M_\nu \in \mathcal{H}_\nu$ be a random matrix with a law invariant under the adjoint action of G_ν . Then $\lambda_\nu(M_\nu)$, conditioned by the fact that the radial part of M_ν is $\lambda \in \mathbb{R}^n$, is uniformly distributed on $GC_\nu(\lambda)$ for $\nu = a, b, d$ and is distributed according to the image of the uniform measure on $GC_c(\lambda)$ by the map $(x^{(1)}, \dots, x^{(2n-1)}, x^{(2n)}) \in GC_c \mapsto (x^{(2)}, x^{(4)}, \dots, x^{(2n-2)}, x^{(2n)}) \in \mathbb{R}^{n(n+1)/2}$, for $\nu = c$.

4 Interlaced determinantal point processes

Considering the eigenvalues of the main minors of some random Hermitian matrices from the classical complex Lie algebras, we construct random configurations on $\mathbb{N} \times \mathbb{R}$ which verify interlacing conditions. We claim that they are determinantal and give their correlation kernels.

Theorem 2. For $\nu = a, b, c$ or d , let $M \in \mathcal{H}_\nu$ be a random matrix with a invariant law invariant under the adjoint action of G_ν . Let ψ_i , $i = 1, \dots, n$, be measurable functions on \mathbb{R} , null on \mathbb{R}_- for $\nu = b, c, d$, such that for all $k \in \mathbb{N}$, $x^k \psi_i(x)$ is integrable on \mathbb{R} . We suppose that the strictly positive eigenvalues (resp. the eigenvalues) of M , $\nu = b, c, d$ (resp. $\nu = a$) have a density with respect to the Lebesgue measure proportional to $\Delta_\nu(x) \det(\psi_j(x_i))_{1 \leq i, j \leq n}$, where $\Delta_a(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$, $\Delta_b(x) = \Delta_c(x) = \prod_{i=1}^n x_i \prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2)$, $\Delta_d(x) = \prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2)$, $x \in \mathbb{R}^n$.

Let us consider the point processes

$$\xi_a = \sum_{i=1}^n \sum_{j=1}^i \delta_{i, \Lambda_{a,j}^{(i)}}, \quad \xi_b = \sum_{i=1}^{2n} \sum_{j=1}^{[(i+1)/2]} \delta_{i, |\Lambda_{b,j}^{(i+1)}|}, \quad \xi_c = \sum_{i=1}^n \sum_{j=1}^i \delta_{i, \Lambda_{c,j}^{(i)}}, \quad \xi_d = \sum_{i=1}^{2n-1} \sum_{j=1}^{[(i+1)/2]} \delta_{i, |\Lambda_{d,j}^{(i+1)}|},$$

where $\Lambda_{\nu,j}^{(i)}$ is the j^{th} component of $\lambda_{\nu}^{(i)}(M_{\nu})$. Then,

- (i) The point process ξ_{ν} is determinantal,
- (ii) Its correlation kernel is

$$R((r, y), (s, z)) = -\frac{1_{s>r}(z-y \wedge z)^{\varphi_{\nu}(r)-\varphi_{\nu}(s)-1}}{(\varphi_{\nu}(r)-\varphi_{\nu}(s)-1)!} + \alpha_{\nu} \sum_{k=1}^n [\psi_k]^{-\varphi_{\nu}(r)}(y) \int \frac{\partial^{\varphi_{\nu}(s)} \Delta_{\nu}}{\partial x_k^{\varphi_{\nu}(s)}}(x_1, \dots, x_{k-1}, z, x_{k+1}, \dots, x_n) \prod_{i \neq k} \psi_i(x_i) dx_i$$

where $[\psi_k]^{-i}(y) = \int_y^{+\infty} \frac{1}{(i-1)!} (x-y)^{i-1} \psi_k(x) dx$ if $i > 0$, $[\psi_k]^0 = \psi_k$, $\alpha_{\nu}^{-1} = \int \Delta_{\nu}(x) \prod_{i=1}^n \psi_i(x_i) dx_i$, and $\varphi_a(r) = \frac{1}{2} \varphi_c(r) = n - r$, $\varphi_b(r) = \varphi_d(r) + 1 = 2n - r$,

(iii) Moreover, when we can write $\Delta_{\nu}(x) = \det(\chi_i(x_j))_{1 \leq i,j \leq n}$ where $(\chi_i)_{i \geq 1}$ is a sequence of functions on \mathbb{R} such that $\chi_i \psi_j$ is integrable on \mathbb{R} and $\int_{\mathbb{R}} \chi_i(x) \psi_j(x) dx = \delta_{ij}$, $i, j = 1, \dots, n$, then

$$R_{\nu}((r, y), (s, z)) = -\frac{1_{s>r}(z-y \wedge z)^{\varphi_{\nu}(r)-\varphi_{\nu}(s)-1}}{(\varphi_{\nu}(r)-\varphi_{\nu}(s)-1)!} + \sum_{k=1}^n [\psi_k]^{-\varphi_{\nu}(r)}(y) \frac{d^{\varphi_{\nu}(s)} \chi_k}{dx^{\varphi_{\nu}(s)}}(z).$$

Let us describe some applications of this theorem:

The Gaussian case For $\nu = a, b, c, d$, let $M_{\nu} \in \mathcal{H}_{\nu}$ be a random Gaussian matrix distributed according to a probability measure proportional to $e^{-t_{\nu} \text{Tr}(H^2)} \mu_{\nu}(dH)$ where μ_{ν} is the Lebesgue measure on \mathcal{H}_{ν} , $t_{\nu} = \frac{1}{2}$ for $\nu = b, c, d$ and $t_a = 1$. The matrices M_{ν} , $\nu = a, d, b, c$, satisfy the hypothesis of the Theorem by taking respectively $\nu = a$ and $\psi_i(x) = x^{i-1} e^{-x^2}$, $\nu = d$ and $\psi_i(x) = x^{2i-2} e^{-x^2} 1_{x>0}$, and $\nu = b, c$ and $\psi_i(x) = x^{2i-1} e^{-x^2} 1_{x>0}$. Besides, the hypothesis of the point (iii) are satisfied if we chose $\chi_i = h_{i-1}$ for $\nu = a$, $\chi_i = h_{2i-1}$, for $\nu = b, c$, and $\chi_i = h_{2i-2}$, for $\nu = d$, where $(h_i)_{i \geq 0}$ is the sequence of Hermite normalised polynomials such that h_i has degree i .

GUE and LUE Ensembles The Gaussian, Laguerre and Jacobi unitary ensembles are obtained by taking $\nu = a$, $\psi_i(x) = x^{i-1} e^{-\alpha x^2}$, $\psi_i(x) = x^{i-1} x^{\alpha} e^{-\beta x} 1_{x>0}$, and $\psi_i(x) = x^{i-1} x^{\alpha} (1-x)^{\beta} 1_{0<x<1}$.

If the radial part of M is deterministic and equal to $\lambda \in \mathbb{R}^n$, the theorem remains true up to slight modifications, replacing $\psi_i(x) dx$ by $\delta_{|\lambda_i|}(dx)$ in the kernel R . As we have seen in the example above, (iii) of the Theorem 2 generalises Theorem 1.3 of (7). Let us give a similar result for the orthogonal case.

corollary 1. Let M be distributed according to a probability measure proportional to $e^{-\frac{1}{2} \text{Tr}(H^2)} dH$, where dH is the Lebesgue measure on the anti-symmetric Hermitian matrices. Consider the random vectors $\Lambda^{(i)} \in \mathbb{R}^{[(i+1)/2]}$ of strictly positive eigenvalues of its main minor of order $i+1$, $i \in \mathbb{N}^*$. Then the point process $\sum_{i=1}^{+\infty} \sum_{j=1}^{[(i+1)/2]} \delta_{i, \Lambda_j^{(i)}}$ is determinantal on $\mathbb{N}^* \times \mathbb{R}_+$ with correlation kernel

$$R((r, y), (s, z)) = -\frac{1_{r \leq s}}{(s-r-1)!} (z-y)^{s-r-1} 1_{y < z} + \sum_{i=0}^{\lceil \frac{r+1}{2} \rceil \wedge \lceil \frac{s+1}{2} \rceil} \frac{(2^r (r-2i+1)!)^{1/2}}{(2^s (s-2i+1)!)^{1/2}} h_{s-2i+1}(z) h_{r-2i+1}(y) e^{-y^2} + \sum_{i=\lceil \frac{r+1}{2} \rceil + 1}^{\lceil \frac{s+1}{2} \rceil} \frac{h_{s-2i+1}(z)}{(2^{s-2i+1} (s-2i+1)!)^{1/2}} \int_y^{+\infty} \frac{(x-y)^{2i-r-2}}{(2i-r-2)!} e^{-x^2} dx$$

For types A and C , the proof of Theorem 2 rests on the criterion of lemma 3.4 in (3), adapted for a continuous framework. For the orthogonal cases B and D , the criterion needs to be slightly modified. For instance, for the odd orthogonal case, we show that:

Proposition 2. Let $\phi_{2r-1} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\phi_{2r-2} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and $\psi_r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $r = 1, \dots, n$, be measurable functions. Let $X^{(2i-1)} \in \mathbb{R}_+^i$ and $X^{(2i)} \in \mathbb{R}_+^i$, $i = 1, \dots, n$, be $2n$ random variables. Suppose that $(X^{(1)}, \dots, X^{(2n)})$ has a density with respect to the Lebesgue measure proportional to

$$\prod_{r=1}^n [\det(\phi_{2r-2}(x_i^{(2r-2)}, x_j^{(2r-1)}))_{i,j \leq r} \det(\phi_{2r-1}(x_i^{(2r-1)}, x_j^{(2r)}))_{i,j \leq r} \det(\psi_j(x_i^{(2n)}))_{1 \leq i,j \leq n},$$

where $x_r^{2r-2} \in \mathbb{R}$, $r = 1, \dots, n$ are arbitrary fixed real numbers. Let us write

$$\phi_r * \phi_s(x, y) = \int_0^\infty \phi_r(x, z) \phi_s(z, y) dz, \quad \phi^{(r,s)} = \begin{cases} \phi_r * \phi_{r+1} * \dots * \phi_{s-1} & \text{if } r < s \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_{r-1} * \phi^{(r,s)} = \begin{cases} \phi_{r-1} * \phi^{(r,s)} & \text{if } r < s \\ \phi_{r-1} & \text{if } r = s \end{cases}, \quad \psi_{r-k}^r(x) = \begin{cases} \int_0^\infty \phi^{(r,2n)}(x, y) \psi_k(y) dy & \text{if } r < 2n. \\ \psi_k(x) & \text{if } r = 2n \end{cases}$$

Let $M = (M_{ij})_{1 \leq i, j \leq n}$ defined by $M_{ij} = \int_0^\infty \phi_{2i-2} * \phi^{(2i-1, 2n)}(x_i^{2i-2}, x) \psi_j(x) dx$. Suppose that all these integrals are finite. Then M is invertible and the point process $\sum_{i=1}^{2n} \sum_{j=1}^{[(i+1)/2]} \delta_{i, X_j^{(i)}}$ is determinantal on $\{1, \dots, 2n\} \times \mathbb{R}_+$ with correlation kernel

$$K((r, x), (s, y)) = -\phi^{(r,s)}(x, y) + \sum_{k=1}^n \psi_{r-k}^r(x) \sum_{l=1}^{[(s+1)/2]} (M^{-1})_{kl} \phi_{2l-2} * \phi^{(2l-1, s)}(x_l^{2l-2}, y).$$

Remark 1. The point process ξ_b defined at the Theorem 2 is obtained by taking $x_r^{2r-2} = 0$, $r = 1, \dots, n$, and $\phi_r(x, y) = 1_{y \geq x}$, for $x, y \in \mathbb{R}_+$, $r = 1, \dots, 2n$.

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